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Vortex surface method: some numerical problems of the potential calculation

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SUMMARY

The singularities method is used to analyze the flow around an isolated profile or through a plane cascade. In this paper, a numerical study has been developed in order to discuss the accuracy of solutions. The aims of this study are summarized as follows: (1) to expose the elements that influence the method—precision in the geometrical profile definition, trailing-edge geometry, smoothing problems, number of discretization points, precision of calculation, etc.; (2) to provide an accurate solution for these different problems. For example, some profiles, obtained by the Joukowski transformation, present, in spite of an analytical definition, a crossing of the suction and pressure sides at the trailing edge. This crossing causes a serious error in the velocity field computation. A new procedure to solve this problem is presented. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: matrix stability; singularity method; solution accuracy

1. INTRODUCTION

In recent years, the development of numerical calculation has led to reductions in calculation times and costs and has enabled a deeper understanding of the flow structure through turbomachinery and physical processes that govern the internal phenomena. The singularities method is used to analyze the flow through the turbomachinery. This method enables a particular solution of the Laplace equation, which satisfies the boundary conditions imposed. The method is very useful because it allows one, once the calculation programs are ready, to study the flows by the superposition of the following elementary flows: a basic uniform flow and sources, sinks, or vortices located at well-chosen points in the flow field. Using this principle, Korn represented the solution of the non-rotational flow around a body by the superposition of a uniform flow and a set of sources and doublets. Prager and Kellogg used a continuous distribution of vortices to solve the same problem. Then, many methods, whose

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complexity varied with the kind of singularities representation used, were developed. Martensen was one of the original investigators of this method, computing the flow through a cascade by using a discrete distribution of vortices on the blade surfaces. Smith and Hess, and Giesing and Bristow adopted a continuous distribution of sources. Using the work of the latter, Bhately, Stevens *et al.*, and Mavrilis replaced the continuous distribution of sources by a continuous distribution of vortices. Minassian applied the method for the calculation of the compressible flow through a tandem cascade using a continuous distribution of vortices. McFarland extended the Giesing method, including a distribution of sources and doublet, to compute the quasi-three-dimensional and compressible flow. In spite of the emergence of other more developed but slower computation methods taking into account the viscous and turbulence effects, the singularities method makes use of the prototype two-dimensional blade to blade procedure, taking advantage of its precision and rapidity, and has today many users. Following the progress made in numerical calculation means and memory capacity, the applications of the method were extended to three-dimensional cases (Spyros and Gasser). Specially adapted to solve unsteady flow and cavitational problems, the method is used to study vorticity shedding, the interaction between profile and vortices, and also cavitation. The method was also applied, in the case of the inverse design, where one determines the profile geometry corresponding to a well-defined pressure or velocity distribution. Synthesis works, published in 1975 by Hess [1] and in 1989 by Sarpkaya [2] on the method of calculation of flow using the vortices concept (comprising about 600 publications), form a large bibliography in the domain.

A large part of this work is based on the Joukowski profiles for which the analytical solution of the flow is known. We adopt the following nomenclature to define these profiles:

Jouko 04 80 10
Jouko: Joukowski Profile
04: absolute value of the camber angle β
80: $[1 - \text{relative thickness}] \times 100$ ($e = 20$ per cent)
10: angle of attack α_0

The Joukowski profile, with its fine or cusped trailing edge, presents the most difficulties for application of the Kutta condition and consequently on the numerical stability. This profile is presented as an ideal mean to validate a potential calculation code. The aim of this study is to highlight the problems met during the numerical programming of the method and also to propose, in many cases, a solution. We present, in brief, the principle of the singularities method and apply it to calculate the flow around an isolated profile.

2. SINGULARITIES METHOD

The method was developed with the following assumptions:

- The fluid is inviscid and incompressible.
- The absolute flow is non-rotational, steady, and two-dimensional.
- The singularities chosen are vortices.

For a set of vortices distributed on the surface of the profile, we derive the complex conjugate velocity C' from the classical relations

$$C' = u - iv = \frac{dF}{dz} = \frac{i}{2\pi}\sum_{j=1}^{k}\frac{\Gamma_j}{(z-z_j)} \pm \mu\frac{\Gamma_m}{2\cdot\Delta s}\exp(-i\cdot\delta) \tag{1}$$

$F(z)$ is the complex potential induced by the set of vortices Γ_j on the point z (Belamri [3]) (Figures 1 and 2).

The flow around the profile is modeled by the superposition of singularities and a uniform flow in the α_0-direction and with modulus C_0. The vortex distribution checks the zero normal velocity condition or slip condition on the profile surface. The potential flow around a cylinder with circulation allows an infinite number of solutions. After conformal mapping, the same is true for the profile. Only one of these solutions verifies the pressure continuity at the trailing

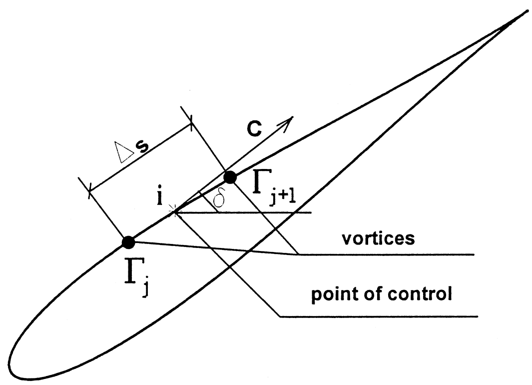


Figure 1. Discretization parameters.

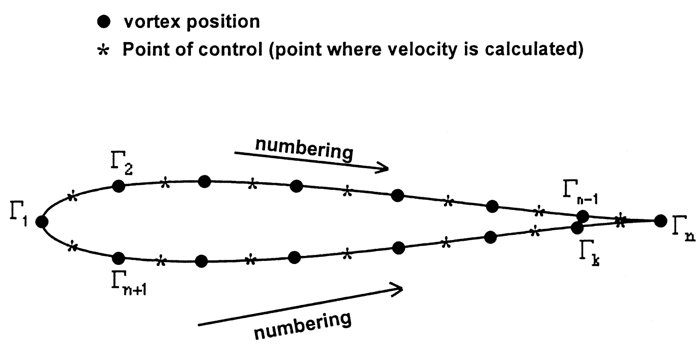


Figure 2. Discretization of the profile.

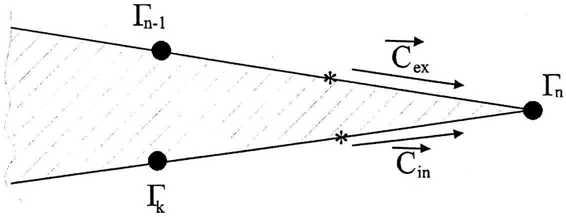


Figure 3. Kutta condition.

edge. The others are singular at this point (infinite velocity, discontinuous pressure). The Kutta condition allows us to choose between these solutions, the one that is obtained by the passage to the limit of the viscous and non-separated flow, when the viscosity tends to zero. So the Kutta condition, for the steady flow case, is verified by equating the velocity modulus on the suction and pressure sides at the trailing edge (Figure 3). The Kutta condition is written as the last equation of the linear system shown previously:

$$C_{\mathrm{ex}} = C_{\mathrm{in}}$$

3. NUMERICAL STUDY

The flexibility of the method and accuracy of results depend mainly on the profile geometry (precision of the surface smoothing, trailing-edge geometry, etc.). In the following sections, these aspects are presented and accurate solutions are provided for the different problems.

3.1. Profile geometrical defect

The linear system matrix elements depend directly on the profile geometry. Two problems are faced.

3.1.1. Influence of the surface joining. Hess [1] showed the sensitivity of the solution to the profile surface joining. In the same perspective, we have calculated the velocity field on the NACA 65 profile surface. This profile, discretized on 400 segments, presents a tangential joining defect (Figure 4(a) and (b)).

3.1.2. The airfoil's surface roughness. Generally, the profiles are obtained from experimental data. The sensitivity of the method to a geometrical defect on the profile surface has a direct consequence on the velocity field, as shown in Figure 4(c) and (d), where we present the velocity field on the symmetrical Fx profile surface [4]. This is irregular at the trailing edge later on. A groove on the profile surface, practically invisible to the naked eye, induces velocity fluctuations and a strong local velocity gradient creating, during boundary layer calculations, unrealistic flow separation. To illustrate this Figure 4(e) and (f) shows the influence of the surface roughness on the velocity field, calculated at the surface of a turbine blade, discretized into 600 segments.

In order to solve this problem, smoothing of the profile is adopted. Generally the smoothing comprises two functions defining, respectively, the suction and the pressure sides of the profile, joined by two circles at the leading and trailing edges. The smoothing, using two polynomial functions whose power depends on the minimization of the standard deviation between the exact and interpolated values, gives a regular velocity field (Figure 4(g)). Here, a polynomial of eight power is used on the suction side and a polynomial of the tenth power on the pressure side.

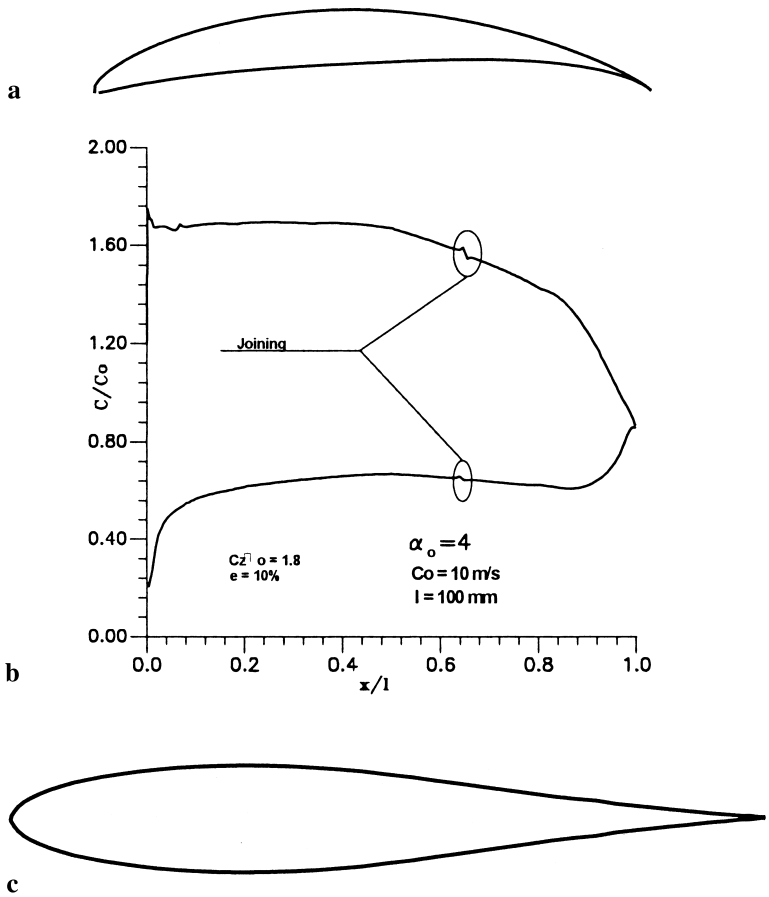


Figure 4. (a) NACA 65 18 10 profile; (b) velocity field on the NACA 65 profile; (c) symmetrical Fx profile; (d) velocity field on the Fx surface; (e) turbine profile; (f) velocity field; (g) after smoothing.

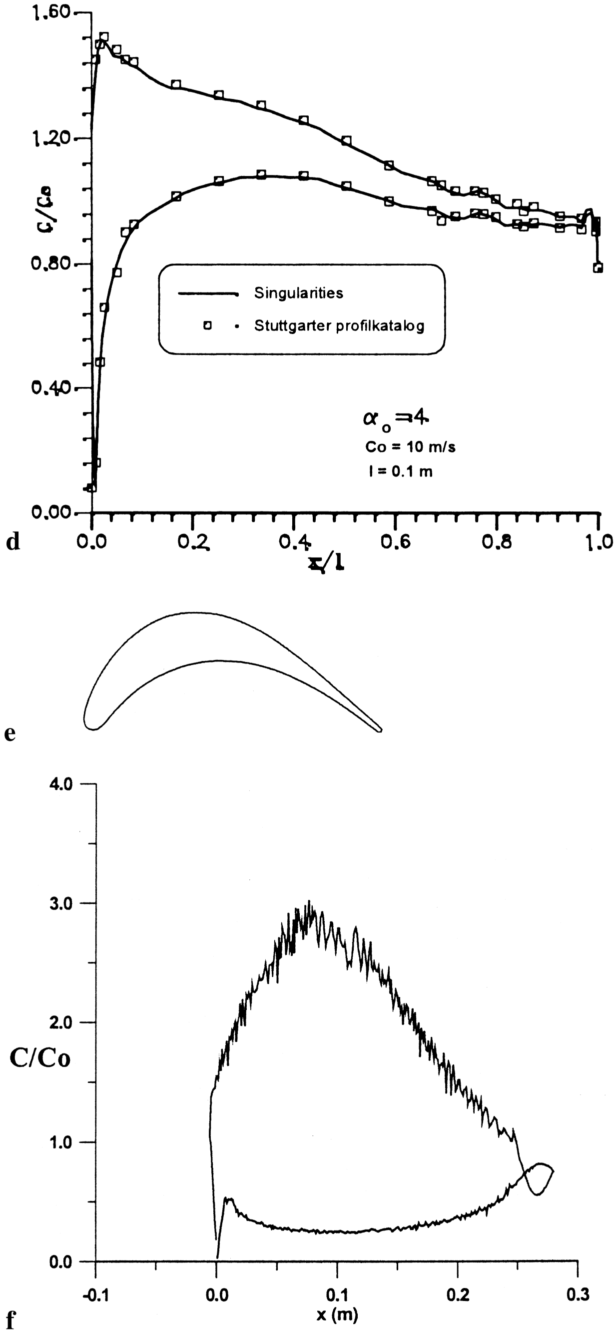


Figure 4 (*Continued*)

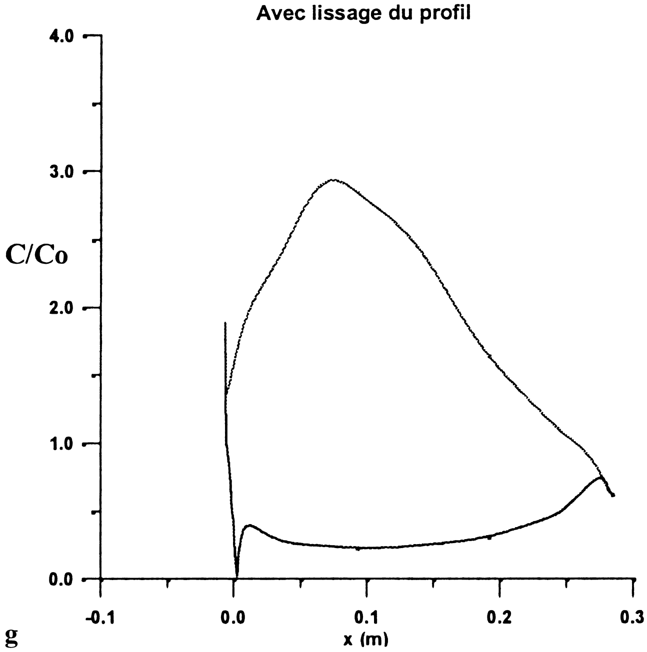


Figure 4 (*Continued*)

3.2. Geometrical defect at the trailing edge related to the conformal mapping

For the Joukowski profiles, in spite of double precision calculation for 600 discretization points, there is a crossing of the suction and pressure sides at the trailing edge. This instability, visible only with a large zoom, leads to a very serious error in the obtained field, even when crossing takes place at a very small distance (d) from the trailing edge, as shown in Figure 5(a)–(c).

In order to solve the problem mentioned above, smoothing is used at the trailing edge. Once this is done, the Kutta condition is employed at the last segment on the trailing edge. A very acceptable velocity field has been obtained (Figure 5(d)). Table I confirms the result.

3.3. Trailing-edge geometry of a thin profile

The influence of a thinner trailing edge on calculation results has been the subject of many studies [5,6]. For our part, we have shown that in spite of trailing-edge smoothing, some irregularities still exist, and furthermore, that they are linked to profile geometry accuracy. The discretization of the latter induces circulation Γ instability (Figure 6(a)), leading to a velocity fluctuation (Figure 6(b)). Indeed the more the profile is discretized, the smaller the last two segments become, tending towards zero, and thus rendering the influence matrix more unstable (no strong main diagonal).

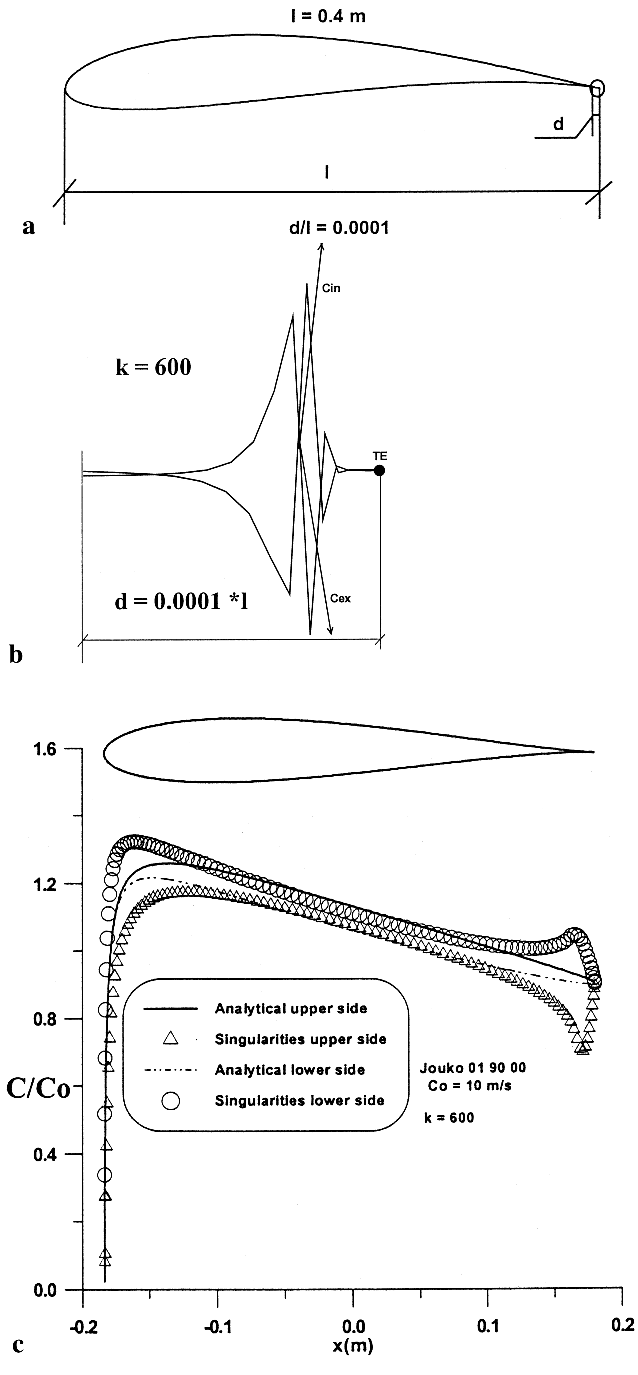


Figure 5. (a) Joukowski profile global view; (b) Joukowski profile—zoom at the trailing edge; (c) velocity field before smoothing the trailing edge; (d) velocity field after smoothing the trailing edge.

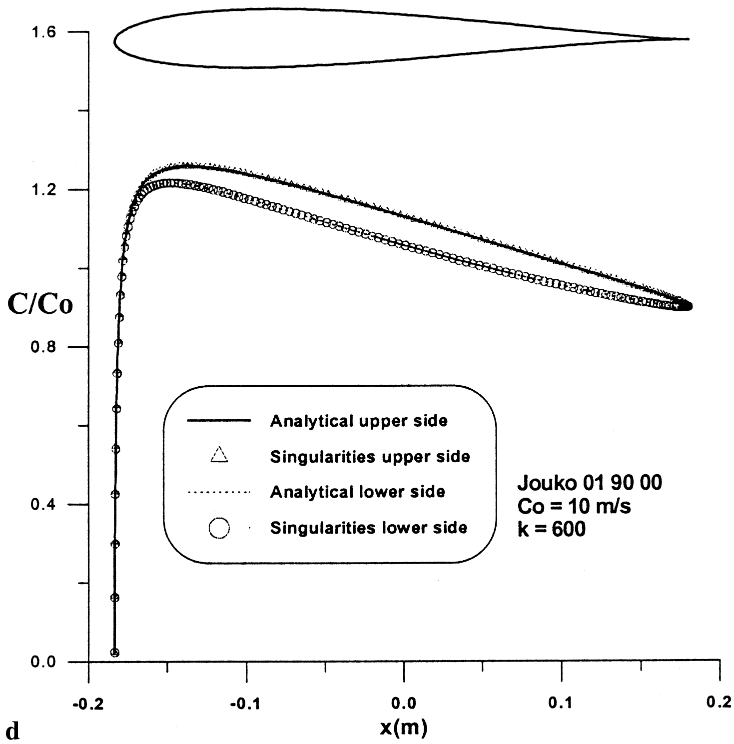


Figure 5 (*Continued*)

Table I. Comparison of total calculated and theoretical circulation.

Γ_{th}	0.2193
Γ_{total} (before smoothing the trailing edge)	−0.8105
Γ_{total} (after smoothing the trailing edge)	0.2192

To solve this problem, Girardi and Bizarro [7] modified the method developed by Hess and Smith by using a weighting function to define the circulation around the Joukowski profile and over a plane cascade. Here, a circulation weighting function has been adopted, as defined in Figure 6(a).

Upstream of the fluctuation (at points A and B located at 95% of the chord) and until the trailing edge (point TE), the circulation is smoothed by the following parabolic function:

$$\Gamma(x) = a \cdot x^2 + b \cdot x + c$$

where a, b, and c are constants defined according to the following conditions:

– the circulation is zero at the trailing edge in accordance with the Kutta condition (Equation (1));
– upstream of the fluctuation, joining is carried out by a common tangent (Equation (2)).

After smoothing the circulation, a very satisfactory result for total circulation with a relative error less than 0.5 per cent is obtained (Figure 6(c)). Figure 6(d) shows one comparison between calculated and theoretical circulation for varying relative thickness (e per cent) and camber (β).

In this application and with circulation smoothing, the problems cited by several authors [8,9] concerning the low relative thickness are unfounded.

3.4. Influence of the number of points

The number of points defines the number of segments into which the profile is discretized. The higher this number, the more precise the profile [10–13]. All the fore-mentioned recommendations about the distribution of singularities have, for objectives, to return to a matrix with a strong main diagonal. Adopting an equal step between the vortex positions, the velocity field

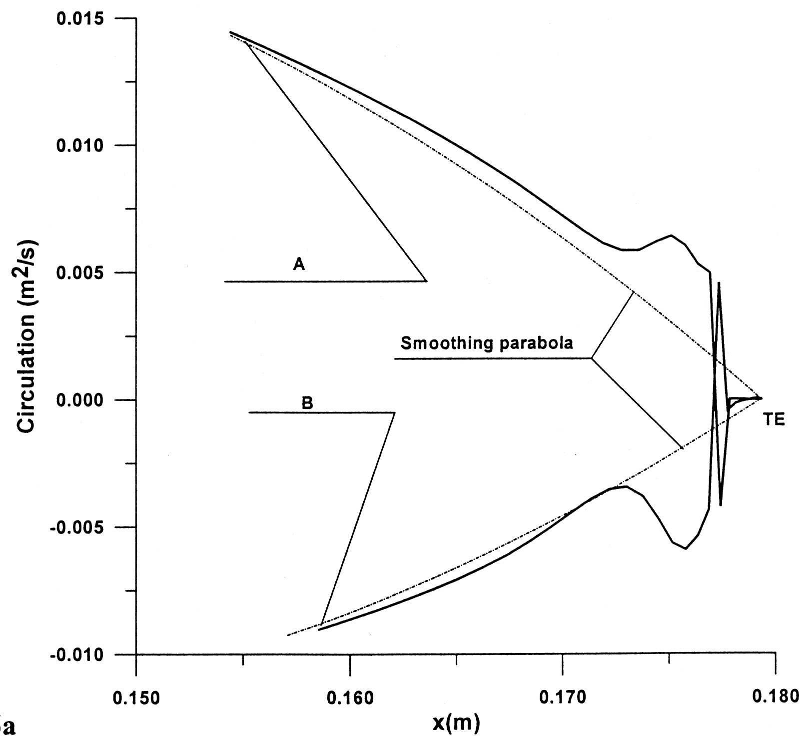
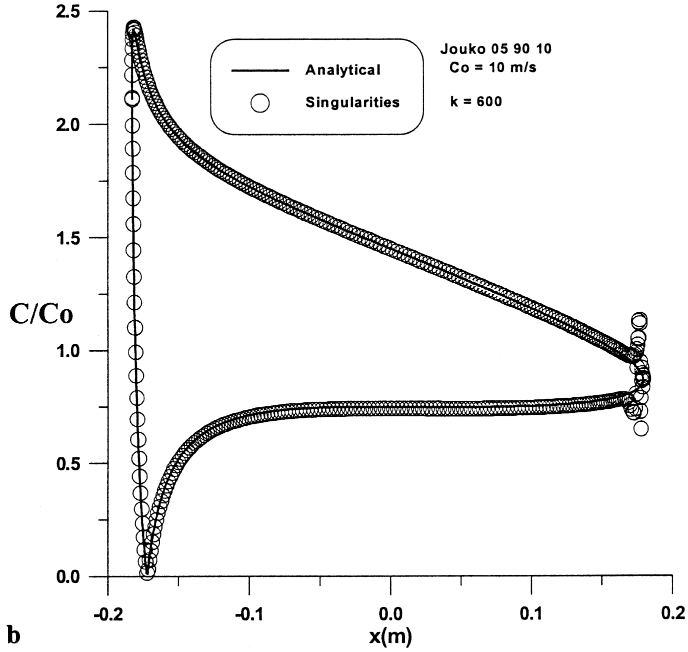
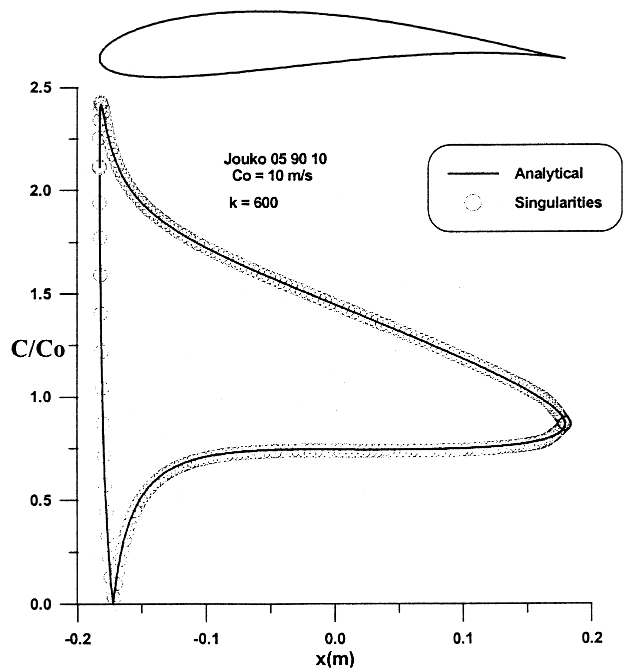


Figure 6. (a) Circulation at the trailing edge; (b) velocity field on the surface of the Joukowski profile; (c) velocity field; (d) comparison of circulation for varying relative thickness and cambers.

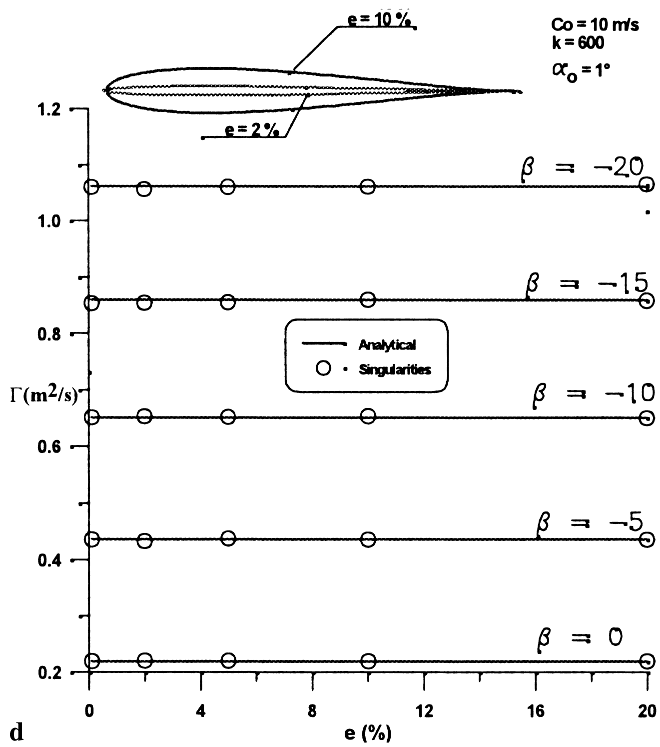


b



c

Figure 6 (*Continued*)



d

Figure 6 (*Continued*)

evolution, as a function of the number of segments discretizing the profile, is presented (Figure 7). Table II indicates the error between the calculated and the theoretical circulation.

3.5. Precision and method of calculation

The choice of whether to conduct calculations using simple or double precision is of great importance for the quality of results [14]. Using the same principle, the velocity field, discretized into 600 segments around a Joukowski profile, was studied using simple and double precision. Figure 8(a) and (b) illustrates the sensitivity of the method to the truncation error.

The considerable difference between the two sets of results can be explained by considering the structure of the matrix A_{ij}. Circulations Γ_j and subsequently velocities C_i are calculated by way of this matrix. It is necessary, in some cases, to have a well-conditioned matrix before calculating the velocity field. In order to highlight the influence of the method of calculation on the obtained result, the velocity field has been calculated around the Joukowski profile using two methods. The first, that of Gauss–Jordan, is direct, more accurate, and uses less computing time. Computing time using a direct solution is proportional to k^3. This time is independent of the number of calculated right sides. The second method is the iterative conjugate gradient method for which the computing time for each right side is proportional to (iter $\times k^2$) where iter is the number of iterations needed for convergence. In the case of simple precision, where the matrix is unstable (Figure 9(a)), the iterative method is quite clearly more accurate than the direct method. For double precision, however (Figure 9(b)), it is preferable to use the direct method, as it requires less computing time—about 1 min on a Pentium PC, 166 MHz with 32 M of RAM.

4. CONCLUSIONS

This study has highlighted the suitability of the singularities method for calculation of the flow around a profile, in comparison with analytical results, and moreover its sensitivity to various numerical parameters.

Of utmost importance is the accuracy with which the profile geometry is defined. Smoothing the surface is essential for profiles defined by experimental data, this being generally the case. An after-treatment of the trailing-edge geometry, based upon current visualization technology, is required in order to correctly apply the Kutta condition. For the general case and during the study of the flow over a profile using the singularities method, the following procedure is recommended in order to gain accurate solutions:

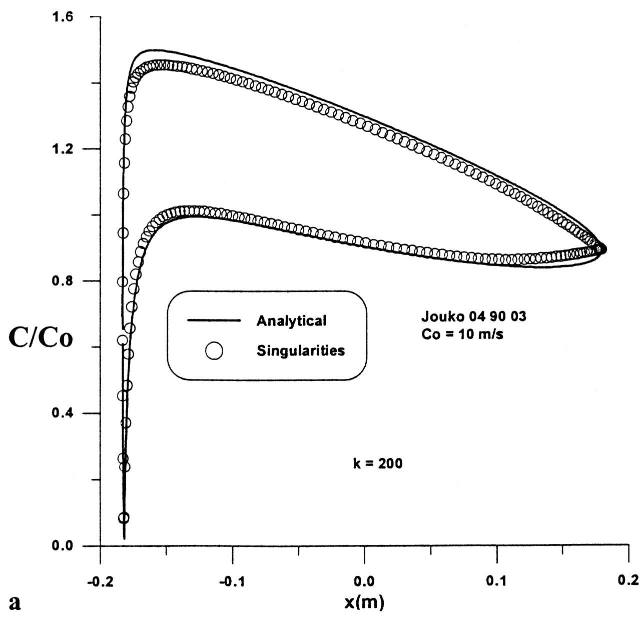


Figure 7. Influence of the number of points.

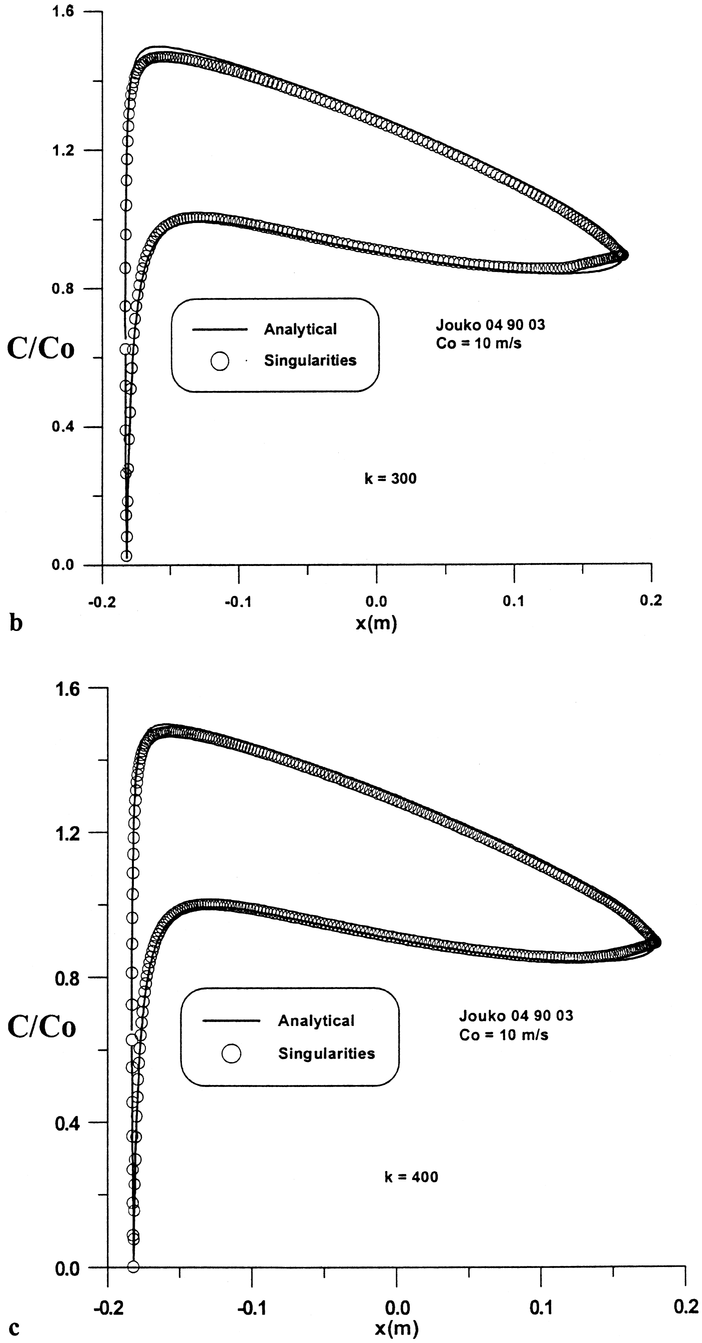


Figure 7 (*Continued*)

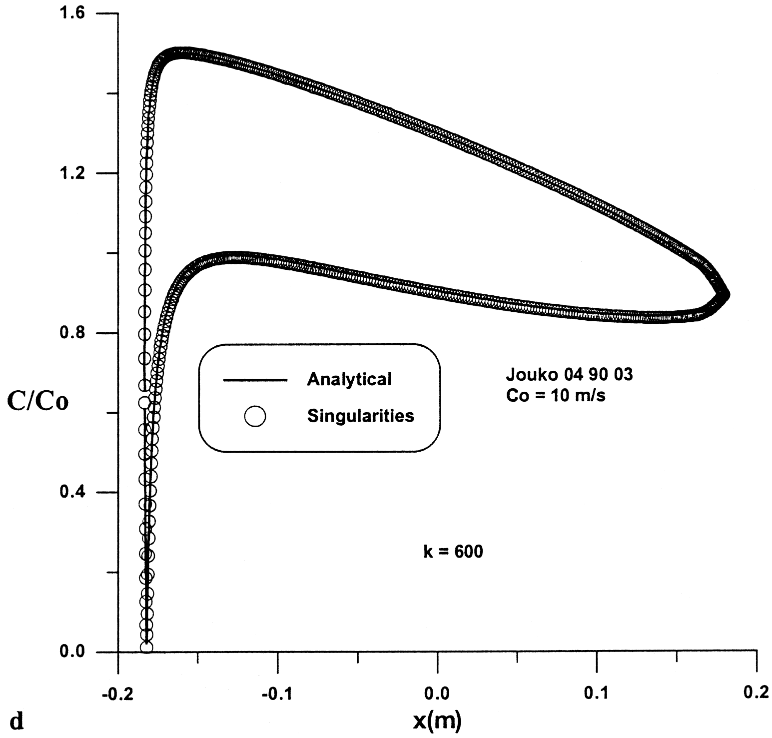


Figure 7 (*Continued*)

Table II. Error in total circulation.

k	200	300	400	600
Error (%)	7.7	2.76	1.02	0.1

- Study of the trailing-edge geometry. If this indicates instability (crossing of the suction and pressure sides), smoothing is necessary.
- After having smoothed the trailing edge, circulation smoothing is essential in certain cases, for example, for profiles with a cusped trailing edge.
- Lastly, recent compilers, using the RAM, have allowed vast increases in the number of discretization points, giving even more accurate solutions.

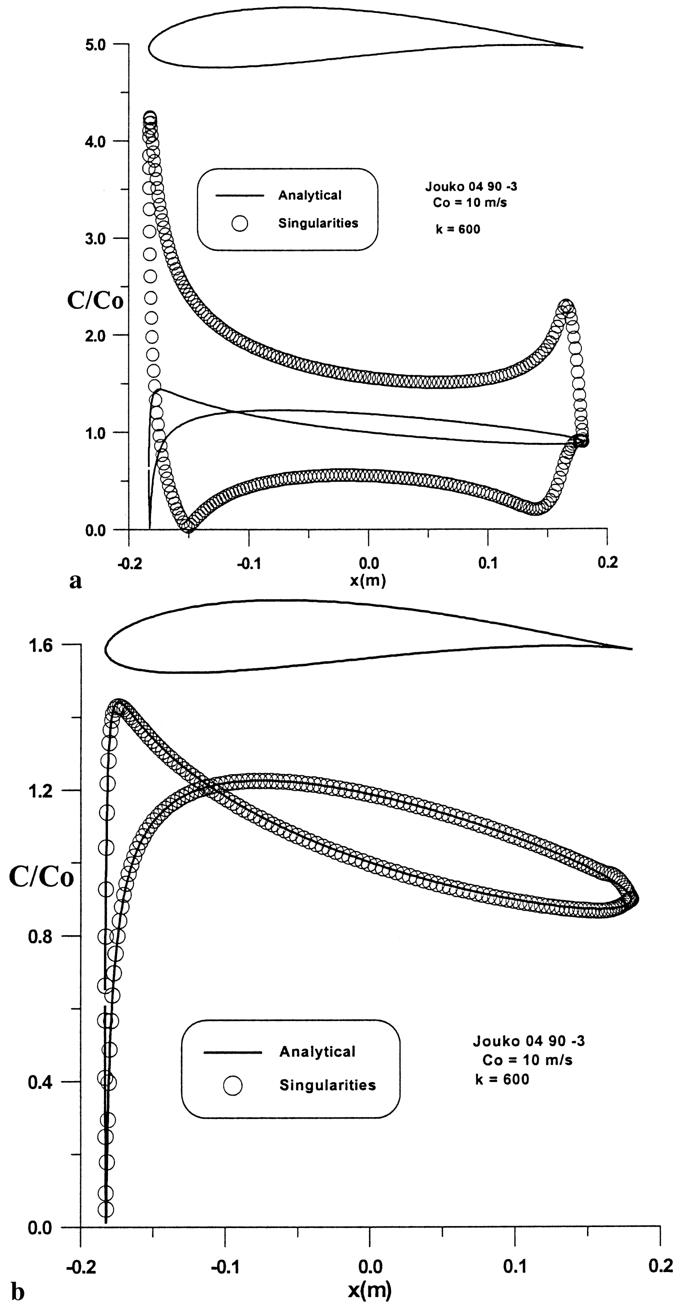


Figure 8. (a) Simple precision; (b) double precision.

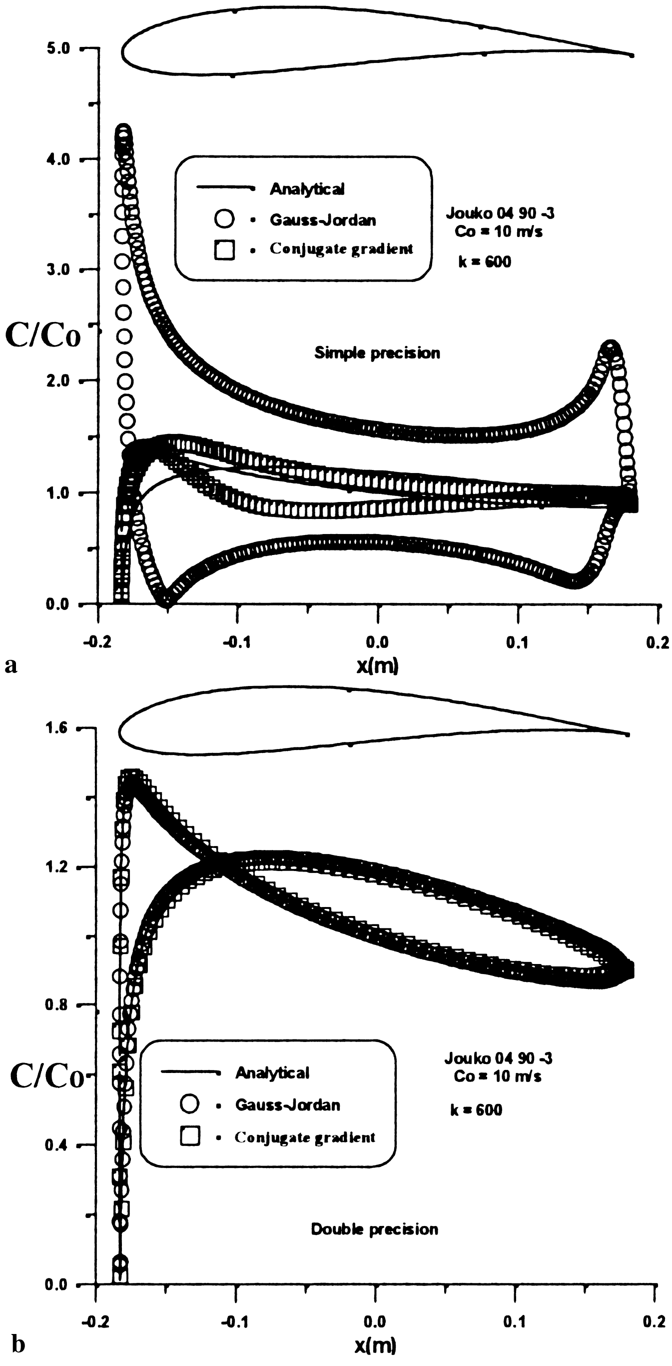


Figure 9. (a) Simple precision; (b) double precision.

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